or

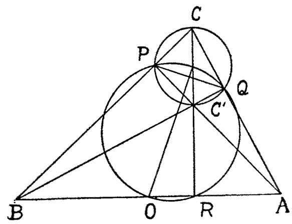
$$X^2 + Y^2 - \frac{b^2 - 2a^2}{k} Y = a^2.$$

This equation represents the locus of R.

II. Solution by Arthur Pelletier, Montreal, Canada.

We have given AB in length only and the fixed points P and Q.

(*a*) O, the middle point of AB, is determined, for $OP = OQ = AB/2$. In the circle $AQPB$, $\angle ACB$ is measured by $\frac{1}{2}(\text{arc}AB - \text{arc } PQ)$ and is therefore constant. Hence the locus of the vertex C, opposite the base AB, is the arc of the circle PCQ.



(*b*) Let R be the foot of the altitude on the base AB, and let C' be the intersection of the three altitudes. In the circle $ACPR$, with diameter AC, $\angle CAP = \angle CRP$. In the circle $AQC'R$, with diameter AC', $\angle QAC' = \angle QRC'$. Finally in the fixed circle $AQPB$, already mentioned, $\angle QAP$, measured by $\frac{1}{2}$ arc PQ, is a constant. Therefore $\angle PRQ$, the sum of the other two angles, is a constant and the locus of R is the arc of the circle PRQ.

As a second point, symmetrical to O with respect to the line PQ, may be the mid-point of AB, it follows that two arcs symmetrical to PCQ and PRQ belong also to the required loci.

Note by Otto Dunkel, Washington University—The triangle $AC'B$ also satisfies the conditions of the problem and the entire circle PCQ is included in the locus under (*a*). Likewise, the entire circle PRQ is the locus of R. If AB rotates about O, C and C' will trace the circle PCQ and change places with the rotation of AB through 180°. It should be noted that the figure takes a somewhat different form when AB intersects PQ.

However, this problem admits of an easier solution by aid of the nine-point circle which is the same for all the triangles and is the locus desired in (*b*); and then it is easy to show that the locus of C is a circle through P and Q with its center at the extremity of the diameter of the nine-point circle that has its other extremity at O.

2902 [1921, 277]. Proposed by C. N. MILLS, Tiffin, Ohio.

Find the locus of a point the feet of perpendiculars from which, on the sides of a triangle, lie on a straight line.

Note by the Editors: It is well known that the circumscribed circle of the triangle is at least part of the locus, by virtue of the theorem of William Wallace, *Mathematical Repository*, March, 1799. No proof of this result is called for in this problem.

I. Solution by R. M. Mathews, Wesleyan University.

With reference to rectangular axes, let the vertices of the triangle be $O \equiv (0, 0)$, $A \equiv (a, 0)$ and $B \equiv (b, c)$. Let $P \equiv (x', y')$ be any point from which perpendiculars PR, PS, PT are dropped to the sides OA, AB, BO, respectively.

Write the equations of the sides of the triangle, then the equations of the perpendiculars and so find the coördinates of the feet as

$$\begin{aligned} R &\equiv (x', 0),\\ S &\equiv \left(\frac{ac^2 - c(a-b)y' + (a-b)^2x'}{(a-b)^2 + c^2},\ \frac{c^2y' - c(a-b)x' + ac(a-b)}{(a-b)^2 + c^2}\right),\\ T &\equiv \left(\frac{b^2x' + bcy'}{b^2 + c^2},\ \frac{bcx' + c^2y'}{b^2 + c^2}\right). \end{aligned}$$

Applying the necessary and sufficient condition that R, S, and T be collinear, we have

$$\begin{vmatrix} x', & 0, & 1 \\ b^2x' + bcy', & bcx' + c^2y', & b^2 + c^2 \\ ac^2 - c(a-b)y' + (a-b)^2x', & c^2y' - c(a-b)x' + ac(a-b), & (a-b)^2 + c^2 \end{vmatrix} = 0.$$

This gives the equation of condition

$$x'^2 + y'^2 - ax' + \frac{ab - b^2 - c^2}{c} y' = 0.$$

The coördinates of O, A and B satisfy this equation. Therefore this is the circumscribed circle, and the circumscribed circle is the required locus.

II. Solution by William Hoover, Columbus, Ohio, and Otto Dunkel, Washington University.

Take the given triangle, ABC, as the triangle of reference in trilinear coördinates, the side opposite A being $\alpha = 0$; also $(\alpha', \beta', \gamma')$ the point from which the perpendiculars are drawn. Take any point in $\alpha = 0$, as $(0, \beta_1, \gamma_1)$; then the straight line through $(\alpha', \beta', \gamma')$ and $(0, \beta_1, \gamma_1)$ is given by

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ 0 & \beta_1 & \gamma_1 \end{vmatrix} = 0$$

or,

$$\alpha(\beta'\gamma_1 - \gamma'\beta_1) - \beta\alpha'\gamma_1 + \gamma\alpha'\beta_1 = 0.$$

This is perpendicular to $\alpha = 0$ if

$$\beta'\gamma_1 - \gamma'\beta_1 - \alpha'\beta_1 \cos B + \alpha'\gamma_1 \cos C = 0$$

or

$$-(\gamma' + \alpha' \cos B)\beta_1 + (\beta' + \alpha' \cos C)\gamma_1 = 0;$$

so that we can write for the coördinates of the foot of the perpendicular from $(\alpha', \beta', \gamma')$ upon $\alpha = 0$,

$$0, \quad \beta' + \alpha' \cos C, \quad \gamma' + \alpha' \cos B.$$

In the same way we get the feet of the other two perpendiculars.

These three points will be on a straight line if

$$\begin{vmatrix} 0 & \beta' + \alpha' \cos C & \gamma' + \alpha' \cos B \\ \alpha' + \beta' \cos C & 0 & \gamma' + \beta' \cos A \\ \alpha' + \gamma' \cos B & \beta' + \gamma' \cos A & 0 \end{vmatrix} = 0.$$

This reduces to [1]

$$(\alpha' \sin A + \beta' \sin B + \gamma' \sin C)(\beta'\gamma' \sin A + \gamma'\alpha' \sin B + \alpha'\beta' \sin C) = 0.$$

The first factor is equal to the area of the triangle divided by the radius of the circumscribed circle and hence cannot be zero. The second factor put equal to zero gives the equation of the circumscribed circle. Hence the circumscribed circle comprises the entire locus.

A simple proof may be obtained by reversing the reasoning given on page 118 of Salmon's *Conic Sections*, edition 6 (London, 1879).

If we assume Wallace's theorem the above may be regarded as a derivation of the equation of the circumscribed circle; for it is the equation of a conic which includes all the points of this circle.

Also solved by T. L. Bennett, A. M. Harding, Arthur Pelletier, and H. L. Olson.

2906 [1921, 277]. Proposed by Elijah Swift, University of Vermont.

Given any number of five digits, reverse the order and subtract the smaller of the two numbers thus formed from the larger. Show that if told the last three digits of this difference, we can find the first two, and give a simple rule for determining them.

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[1] If we multiply the three rows by $\beta'\gamma' \sin A$, $\gamma'\alpha' \sin B$ and $\alpha'\beta' \sin C$, respectively, and add, we shall get a new row whose elements are α', β', and γ' times the expression

$$\beta'\gamma' \sin A + \gamma'\alpha' \sin B + \alpha'\beta' \sin C,$$

which is, therefore, a factor.